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EFFECT OF ELASTIC FOUNDATION ON ASYMMETRIC VIBRATION OF POLAR ORTHOTROPIC LINEARLY TAPERED CIRCULAR PLATES

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Asymmetric vibrations of polar orthotropic circular plates of linearly varying thickness resting on an elastic foundation of *Winkler* type are discussed on the basis of the classical plate theory. *Ritz* method has been employed to obtain the natural frequencies of vibration using the functions based upon the static deflection of polar orthotropic plates, which has faster rate of convergence as compared to the polynomial co-ordinate functions. Frequency parameter of the plate with elastically restrained edge conditions are presented for various values of taper parameter, rigidity ratio and foundation parameter. A comparison of the results with those available in the literature obtained by finite element method, receptance method and polynomial co-ordinate functions shows an excellent agreement.

1. INTRODUCTION

Plates of various geometries i.e. circular, annular, rectangular and polygonal, etc of orthotropic material such as fibre-reinforced composites are extensively used in engineering applications due to their high specific stiffness and light weight characteristics. These plates are widely used in modern aerospace technology. Many structural components in aerospace, mechanical and civil engineering are supported on elastic medium. The vibrational characteristics of plates resting on an elastic medium are different from those of the plates supported only on the boundary. The study of the dynamic response of plates resting on an elastic foundation is of great interest in connection with the reinforced concrete pavements of high runways and the foundations of buildings. A number of papers [1–18] have appeared dealing with natural frequencies of plates of uniform/non-uniform thickness, to investigate the effect of elastic foundation. Gupta *et al.* [4] studied the effect of elastic foundation on axisymmetric vibrations of linearly tapered annular plates using quintic splines technique and that of parabolically tapered annular plates by Chebyshev polynomials. Gupta *et al.* [7] studied the effect of elastic foundation on axisymmetric vibrations of polar orthotropic circular plates of variable thickness by Ritz method. In the recent past, the Ritz method has been applied by research workers to study the plate vibration of different shapes and are reported in [19–25]. Laura *et al.* [9] analyzed the free vibration of a solid circular plate of linearly varying thickness attached to Winkler foundation using the Ritz method. Liew *et al.* [10] employed the differential quadrature method for studying the Mindlin's plate on Winkler foundation. In addition to these, Shih and Blotter [11], Ji Wang [12], Gupta *et al.* [13–17], Arul *et al.* [18] have investigated the

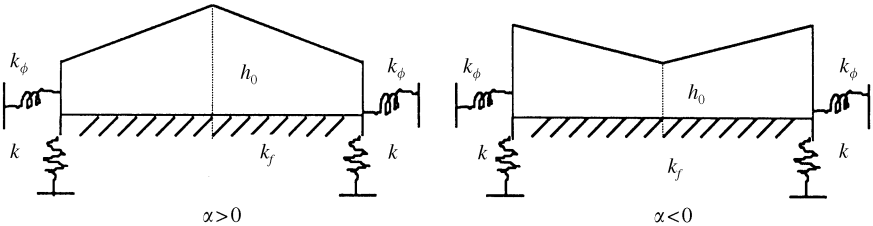

Figure 1.

frequencies of vibration of a plate resting on an elastic foundation. All the above papers deal with axisymmetric vibrations. The analysis of the vibration of plates with elastically restrained edges is an important problem in aeronautical and naval structural engineering. In aircraft structures, the individual plates are connected to the other plates or stiffners at their boundaries and thus have elastic restraint at their edges [26–32].

The present paper analyzes the effect of elastic foundation (Winkler type) on asymmetric vibrations of polar orthotropic circular plates of linearly varying thickness using the Ritz method, where basis functions based upon the static deflection for polar orthotropic plates have been used. The present choice of functions provides a good approximation for deflection and frequencies [8].

2. ENERGY EXPRESSION

Consider a thin circular plate of radius a, thickness $h = h(r)$, resting on *Winkler* foundation of modulus k_f elastically restrained against rotation and translation. Let (r, θ) be the polar co-ordinates of any point on the neutral surface of the plate, referred to as the centre of the plate or origin (Figure 1).

The maximum kinetic energy of the plate is given by

$$T_{max} = \frac{1}{2}\rho\omega^2 \int_0^a \int_0^{2\pi} h\, w^2\, r\, \mathrm{d}\theta\, \mathrm{d}r, \tag{1}$$

where w is the transverse deflection, ρ the mass density, ω the frequency in rad/s. The maximum strain energy of the plate is given by

$$\begin{aligned} U_{max} = \frac{1}{2}\int_0^a \int_0^{2\pi} \Bigg[D_r \left\{ \left(\frac{\partial^2 w}{\partial r^2}\right)^2 + 2\nu_\theta \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\} \\ + D_\theta \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 + D_k \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\}^2 + k_f w^2 \Bigg] r\, \mathrm{d}\theta\, \mathrm{d}r \\ + \frac{1}{2} a k \int_0^{2\pi} w^2(a,\theta)\, \mathrm{d}\theta + \frac{1}{2} a k_\phi \int_0^{2\pi} \left(\frac{\partial w(a,\theta)}{\partial r}\right)^2 \mathrm{d}\theta, \end{aligned} \tag{2}$$

where k and $1/k_\phi$ are the translational and rotational rigidities of the springs and

$$D_r(r) = \frac{E_r h^3}{12(1-\nu_r\nu_\theta)}, \qquad D_\theta(r) = \frac{E_\theta h^3}{12(1-\nu_r\nu_\theta)}, \qquad D_k(r) = \frac{G_{r\theta}h^3}{3}.$$

3. METHOD OF SOLUTION: RITZ METHOD

Ritz method requires that the functional

$$J(w) = U_{max} - T_{max} = \frac{1}{2}\int_0^a \int_0^{2\pi} \left[D_r \left\{ \left(\frac{\partial^2 w}{\partial r^2} \right)^2 + 2\nu_\theta \frac{\partial^2 w}{\partial r^2} \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right) \right\} \right.$$

$$\left. + D_\theta \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right)^2 + D_k \left\{ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} \right) \right\}^2 + k_f w^2 \right] r \,\mathrm{d}\theta\,\mathrm{d}r$$

$$+ \frac{1}{2} a k \int_0^{2\pi} w^2(a,\theta)\,\mathrm{d}\theta + \frac{1}{2} a k_\phi \int_0^{2\pi} \left(\frac{\partial w(a,\theta)}{\partial r} \right)^2 \mathrm{d}\theta - \frac{1}{2}\rho\omega^2 \int_0^a \int_0^{2\pi} h w^2 r \,\mathrm{d}\theta\,\mathrm{d}r \qquad (3)$$

be minimized.

Introducing the non-dimensional variables $\bar{W} = w/a$, $R = r/a$, we assume the deflection function to be

$$\bar{W} = W_a(R)\cos n\theta = \cos n\theta \sum_{i=0}^{m} A_i W_i(R) = \cos n\theta \sum_{i=0}^{m} A_i (1 + \alpha_i R^4 + \beta_i R^{1+p}) R^{2i+n}, \quad (4)$$

where A_i are undetermined coefficients, $p^2 = E_\theta / E_r$ and α_i, β_i are unknown constants to be determined from boundary conditions. The present choice of functions is based upon the static deflection for polar orthotropic circular plates. Using non-dimensional variables $\bar{W}$ and R along with the relation (4) the functional $J(w)$ given by equation (3) becomes

$$J(W_a) = \frac{\pi D_{r0}}{2} \left[\int_0^1 \int_0^{2\pi} \left[\left\{ \left(\frac{\partial^2 W_a}{\partial R^2} \right)^2 + 2\nu_\theta \frac{\partial^2 W_a}{\partial R^2} \left(\frac{1}{R}\frac{\partial W_a}{\partial R} - \frac{n^2 W_a}{R^2} \right) \right\} \right. \right.$$

$$\left. + p^2 \left(\frac{1}{R}\frac{\partial W_a}{\partial R} - \frac{n^2 W_a}{R^2} \right)^2 + n^2 D_{k0} \left\{ \frac{\partial}{\partial R}\left(\frac{W_a}{R} \right) \right\}^2 + K_f W_a^2 \right] R\,\mathrm{d}R$$

$$\left. + K W_a^2(1) + K_\phi \left(\frac{\partial W_a(1)}{\partial R} \right)^2 - \Omega^2 \int_0^1 \int_0^{2\pi} (1 - \alpha R) W_a^2 R\,\mathrm{d}R \right], \qquad (5)$$

where $h = h_0(1 - \alpha R)$ specifies the linear thickness variation, h_0 being the thickness of the plate at the centre, α the taper parameter and

$$D_{r0} = \frac{E_r h_0^3}{12(1 - \nu_r \nu_\theta)}, \qquad D_{k0} = \frac{D_k}{D_{r0}}, \qquad \Omega^2 = \frac{a^4 \omega^2 \rho h_0}{D_{r0}},$$

$$K_f = \frac{a^4 k_f}{D_{r0}}, \qquad K = \frac{a^3 k}{D_{r0}}, \qquad K_\Phi = \frac{a k_\Phi}{D_{r0}}.$$

The minimization of the functional $J(W_a)$ given by equation (5) requires

$$\frac{\partial J(W_a)}{\partial A_i} = 0, \quad i = 0, 1, \ldots, m. \qquad (6)$$

This leads to a system of homogeneous equations in A_i, $i = 0, 1, \ldots, m$ whose non-trivial solution leads to the frequency equation

$$|A - \Omega^2 B| = 0, \qquad (7)$$

where $A = [a_{ij}]$ and $B = [b_{ij}]$ are square matrices of order $m + 1$ and

$$a_{ij} = \int_0^1 (1-\alpha R)^3 \left[W_i'' W_j'' + 2\nu_\theta W_i'' \left(\frac{W_j'}{R} - \frac{n^2 W_j}{R^2} \right) + p^2 \left(\frac{W_i'}{R} - \frac{n^2 W_i}{R^2} \right)\left(\frac{W_j'}{R} - \frac{n^2 W_j}{R^2} \right) \right.$$
$$\left. + n^2 D_{k0} \left(\frac{W_i'}{R} - \frac{W_i}{R^2} \right)\left(\frac{W_j'}{R} - \frac{W_j}{R^2} \right) + K_f W_i W_j \right] R\,\mathrm{d}R$$
$$+ K W_i(1)\, W_j(1) + K_\Phi W_i'(1)\, W_j'(1) \tag{8}$$

and

$$b_{ij} = \int_0^1 (1-\alpha R) W_i W_j R\,\mathrm{d}R. \tag{9}$$

As each deflection function W_i has to satisfy the boundary conditions [33, p. 14], we have

$$K_\phi \frac{\mathrm{d}W_i(1)}{\mathrm{d}R} = -(1-\alpha)^3 \left[\frac{\mathrm{d}^2 W_i}{\mathrm{d}R^2} + \nu_\theta \left(\frac{1}{R}\frac{\mathrm{d}W_i}{\mathrm{d}R} - n^2 \frac{W_i}{R^2} \right) \right]_{R=1}, \tag{10}$$

$$K W_i(1) = (1-\alpha)^3 \left[\frac{\mathrm{d}}{\mathrm{d}R} \left(\frac{\mathrm{d}^2 W_i}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}W_i}{\mathrm{d}R} - n^2 \frac{W_i}{R^2} \right) - \frac{1}{2} D_{k0} \left(\frac{1}{R}\frac{\mathrm{d}W_i}{\mathrm{d}R} - \frac{W_i}{R^2} \right) \right]_{R=1}. \tag{11}$$

Substituting equation (4) in equations (10) and (11), we get

$$\alpha_i = \frac{S_i u_i - s_i U_i}{Q_i s_i - S_i q_i}, \qquad \beta_i = \frac{q_i U_i - Q_i u_i}{Q_i s_i - S_i q_i},$$

where

$$Q_i = K_\phi (2i+n+2) + (1-\alpha)^3 \{(2i+n+2)(2i+n+1) + \nu_\theta (2i+n+2-n^2)\},$$

$$S_i = K_\phi (2i+n+p-1) + (1-\alpha)^3 \{(2i+n+p-1)(2i+n+p-2) + \nu_\theta (2i+n+p-1-n^2)\},$$

$$U_i = K_\phi (2i+n-2) + (1-\alpha)^3 \{(2i+n-2)(2i+n-3) + \nu_\theta (2i+n-2-n^2)\},$$

$$q_i = K - (1-\alpha)^3 \left\{(2i+n+2)(2i+n+1)^2 - (1+n^2)(2i+n+2) + 2n^2 - \frac{D_{k0}}{2}(2i+n+1)\right\},$$

$$s_i = K - (1-\alpha)^3 \left\{(2i+n+p-1)(2i+n+p-2)^2 - (1+n^2)(2i+n+p-1) + 2n^2 - \frac{D_{k0}}{2}(2i+n+p-2)\right\},$$

$$u_i = K - (1-\alpha)^3 \left\{(2i+n-2)(2i+n-3)^2 - (1+n^2)(2i+n-2) + 2n^2 - \frac{D_{k0}}{2}(2i+n-3)\right\}.$$

4. NUMERICAL RESULTS

The frequency equation (7) has been solved by hybrid secant method for the first two modes of vibrations corresponding to $n = 0$–2 to investigate the effect of elastic foundation parameter K_f (= 0·00 (0·01) 0·05) on the natural frequencies for various values of rigidity ratio E_θ/E_r (= 0·5, 0·75, 1·00, 5·00), taper parameter α (= 0, ± 0·3) and flexibility parameter K_ϕ (= 0, 10, $10^{20} \cong \infty$). The case of boundary conditions, i.e., free, simply supported and clamped are obtained by proper choice of K and K_ϕ. The shear modulus D_{k0} and Poisson's ratio ν_θ of the plate have been fixed as 1·4 and 0·3 respectively.

TABLE 1

Frequency parameter Ω as a function of flexibility, taper and orthotropy parameters for circular plate: $\nu_\theta = 0{\cdot}3$, $K = 0$

$K_f = 0{\cdot}01$												
	$\alpha = -0{\cdot}3$						$\alpha = 0{\cdot}3$					
K_ϕ/p^2	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00
	Ω_{01}		Ω_{10}		Ω_{20}		Ω_{01}		Ω_{10}		Ω_{20}	
00	9·0357	9·9051	8·9070	9·7477	10·1343	13·8136	10·9760	12·1000	11·3682	12·4437	12·1111	14·3072
10	9·0482	9·9063	9·4594	10·3753	11·7431	16·7948	11·0419	12·1126	11·5195	12·6313	12·8750	15·9313
10^{20}	9·0503	9·9069	9·6800	10·7344	12·6705	18·8936	11·0457	12·1140	11·5344	12·6550	12·9760	16·1782
	Ω_{02}		Ω_{11}		Ω_{21}		Ω_{02}		Ω_{11}		Ω_{21}	
00	12·2189	21·7626	24·2362	32·5838	39·3425	55·9924	12·9670	18·1344	20·0247	24·6811	29·4504	39·4513
10	16·2997	25·0305	29·8242	37·4765	45·1584	62·2318	15·0204	20·5529	24·4219	29·0380	35·0655	46·1437
10^{20}	18·2949	27·3166	33·6980	41·5600	50·4100	68·9469	15·2759	20·9230	25·0927	29·7921	36·1010	47·4734
$K_f = 0{\cdot}02$												
	Ω_{01}		Ω_{10}		Ω_{20}		Ω_{01}		Ω_{10}		Ω_{20}	
00	12·7534	14·0016	12·5948	13·7842	13·4665	16·8607	15·3865	17·0611	16·0614	17·5860	16·6763	19·0800
10	12·7863	14·0051	13·0143	14·2516	14·7323	19·3903	15·5434	17·0951	16·1324	17·6864	17·1967	20·2858
10^{20}	12·7920	14·0068	13·1855	14·5254	15·4920	21·2439	15·5539	17·0990	16·1394	17·6992	17·2666	20·4731
	Ω_{02}		Ω_{11}		Ω_{21}		Ω_{02}		Ω_{11}		Ω_{21}	
00	15·3338	23·9706	25·9152	34·0900	40·3696	56·8692	17·1872	21·8721	22·8696	27·4962	31·4776	41·2872
10	18·7740	26·9935	31·2128	38·8005	46·0763	63·0253	18·6113	23·8327	26·7670	31·4305	36·7548	47·7067
10^{20}	20·5490	29·1415	34·9430	42·7668	51·2401	69·6691	18·8009	24·1403	27·3726	32·1208	37·7387	48·9886

TABLE 2

Frequency parameter Ω as a function of flexibility, taper and orthotropy parameters for circular plate: $\nu_\theta = 0{\cdot}3$, $K = 10$

	$K_f = 0{\cdot}01$											
	$\alpha = -0{\cdot}3$						$\alpha = 0{\cdot}3$					
K_ϕ/p^2	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00
	Ω_{00}		Ω_{10}		Ω_{20}		Ω_{00}		Ω_{10}		Ω_{20}	
00	9·7232	10·6671	10·4759	11·2347	11·9039	15·5090	13·3530	14·5749	13·7639	14·8817	14·8498	18·2775
10	9·8746	10·6926	10·6809	11·5750	13·0015	17·9374	13·4193	14·5869	13·9351	15·1527	15·7565	20·3892
10^{20}	9·9076	10·7058	10·7675	11·7780	13·6669	19·7148	13·4337	14·5931	14·0079	15·3160	16·3186	21·9781
	Ω_{01}		Ω_{11}		Ω_{21}		Ω_{01}		Ω_{11}		Ω_{21}	
00	13·5908	22·5808	24·8975	33·0971	39·7446	56·2722	16·3718	24·7039	26·5284	34·5771	40·7782	57·1445
10	16·9219	25·5255	30·2148	37·8061	45·4408	62·4386	19·2950	27·4464	31·5837	39·1174	46·3525	63·2294
10^{20}	18·6641	27·6272	33·9160	41·7492	50·5669	69·0596	20·8659	29·4284	35·1519	42·9499	51·3940	69·7805
	$K_f = 0{\cdot}02$											
	Ω_{00}		Ω_{10}		Ω_{20}		Ω_{00}		Ω_{10}		Ω_{20}	
00	11·1275	12·6215	12·8963	14·0165	14·3935	16·8849	15·4168	17·3419	17·0846	18·6667	18·3631	21·0622
10	11·6202	12·8493	12·9210	14·0165	14·5708	17·6721	15·8376	17·5540	17·1198	18·6684	18·4832	21·6699
10^{20}	11·6762	12·8806	12·9239	14·0165	14·5963	17·7975	15·8887	17·5840	17·1240	18·6687	18·5005	21·7671
	Ω_{01}		Ω_{11}		Ω_{21}		Ω_{01}		Ω_{11}		Ω_{21}	
00	15·5727	20·1684	21·8142	26·1721	30·6678	40·3565	19·2533	23·6337	24·5039	28·8746	32·6243	42·1600
10	16·3708	21·6600	25·2398	29·7721	35·6638	46·6078	19·7994	24·8350	27·5359	32·1245	37·3323	48·1587
10^{20}	16·4888	21·9066	25·7847	30·4123	36·5944	47·8470	19·8818	25·0372	28·0264	32·7096	38·2163	49·3532

Table 3

Frequency parameter Ω as a function of flexibility, taper and orthotropy parameters for circular plate: $\nu_\theta = 0{\cdot}3$, $K = 10^{20}$

	$K_f = 0{\cdot}01$											
	$\alpha = -0{\cdot}3$						$\alpha = 0{\cdot}3$					
K_ϕ/p^2	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00	0·50	5·00
	Ω_{00}		Ω_{10}		Ω_{20}		Ω_{00}		Ω_{10}		Ω_{20}	
00	10·4191	14·6542	18·1823	23·4406	29·3023	43·5781	11·2034	13·4338	15·4222	18·1611	21·9648	30·0797
10	12·9034	17·3458	22·0610	27·2231	33·5063	48·1565	12·4444	15·2181	18·6936	21·7140	26·5558	35·7293
10^{20}	14·9266	20·2774	26·5462	32·3360	39·8085	56·3544	12·6935	15·6580	19·5305	22·7248	27·9545	38·1401
	Ω_{01}		Ω_{11}		Ω_{21}		Ω_{01}		Ω_{11}		Ω_{21}	
00	33·3190	47·1553	55·7164	66·6074	78·0891	101·7802	25·3564	34·5263	40·7906	47·5216	56·0747	70·9581
10	37·6803	51·7201	60·3736	71·2990	82·9972	106·5747	30·5628	40·6169	47·4109	54·5416	63·6078	79·0779
10^{20}	44·8201	60·5192	70·2448	82·1452	95·0433	120·5582	32·2471	42·8735	50·0784	57·5769	67·0872	83·4146
	$K_f = 0{\cdot}02$											
	Ω_{00}		Ω_{10}		Ω_{20}		Ω_{00}		Ω_{10}		Ω_{20}	
00	13·9982	17·8464	20·3754	25·5000	30·7040	44·7017	15·4363	17·8149	18·8811	21·7432	24·5651	32·4485
10	15·9649	20·1385	23·9161	29·0281	34·7344	49·1779	16·2975	19·1384	21·5850	24·7388	28·7067	37·7163
10^{20}	17·6705	22·7369	28·1250	33·8870	40·8597	57·2293	16·4766	19·4764	22·3006	25·6163	30·0016	39·5888
	Ω_{01}		Ω_{11}		Ω_{21}		Ω_{01}		Ω_{11}		Ω_{21}	
00	34·5947	48·2299	56·4813	67·3714	78·8239	102·3809	27·5641	36·5223	42·2149	48·9977	57·1007	71·9198
10	40·2968	38·8148	61·0790	72·0130	83·5043	107·0908	32·3909	42·3168	48·6371	55·8272	64·5105	80·0154
10^{20}	45·7822	61·3835	70·8542	82·7679	95·4438	120·9037	33·9574	44·4794	51·2354	58·7904	67·9633	84·2617

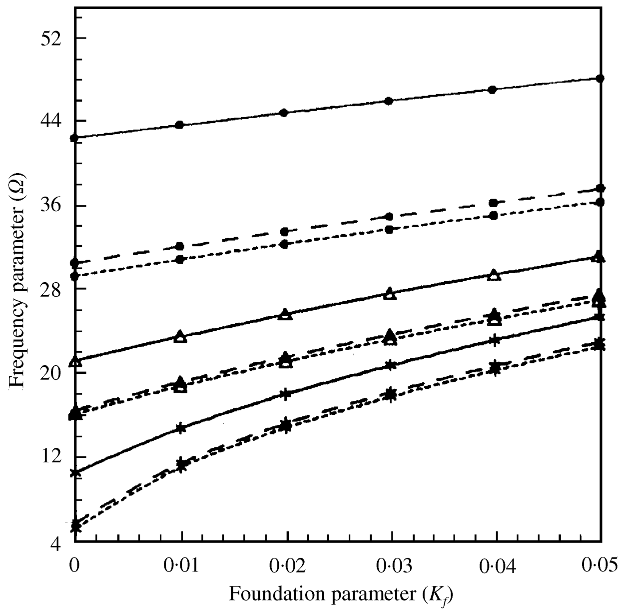

Figure 2. Fundamental frequency parameter for: ....; $E_r/E_\theta = 0{\cdot}75$; -------; $E_r/E_\theta = 1{\cdot}00$, and ———; $E_r/E_\theta = 5{\cdot}00$. Simply supported plate for $\alpha = -0{\cdot}3$: *—*—*—*—*, $n = 0$; △—△—△—△—△, $n = 1$; ●–●–●–●–●, $n = 2$.

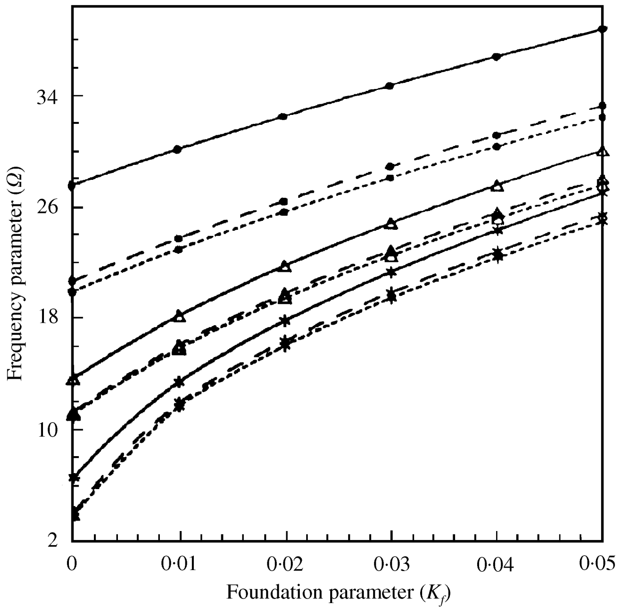

Figure 3. Fundamental frequency parameter for: ....; $E_r/E_\theta = 0{\cdot}75$; -------; $E_r/E_\theta = 1{\cdot}00$, and ———; $E_r/E_\theta = 5{\cdot}00$. Simply supported plate for $\alpha = 0{\cdot}3$: *—*—*—*—*, $n = 0$; △—△—△—△—△, $n = 1$; ●–●–●–●–●, $n = 2$.

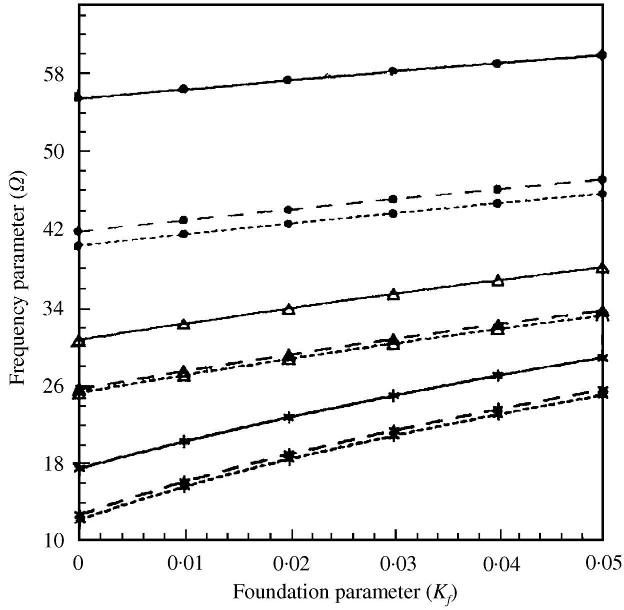

Figure 4. Fundamental frequency parameter for: ....; $E_r/E_\theta = 0{\cdot}75$; -------; $E_r/E_\theta = 1{\cdot}00$, and ———; $E_r/E_\theta = 5{\cdot}00$. Clamped plate for $\alpha = -0{\cdot}3$: ∗—∗—∗—∗—∗, $n = 0$; △—△—△—△—△, $n = 1$; ●–●–●–●–●, $n = 2$.

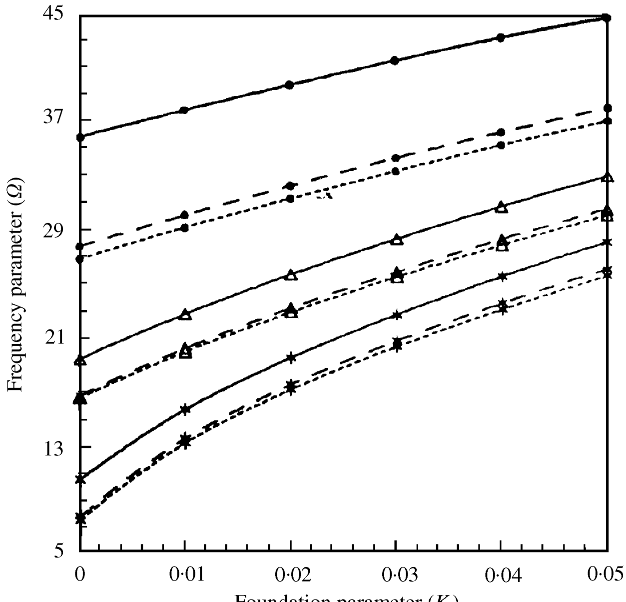

Figure 5. Fundamental frequency parameter for: ....; $E_r/E_\theta = 0{\cdot}75$; -------; $E_r/E_\theta = 1{\cdot}00$, and ———; $E_r/E_\theta = 5{\cdot}00$. Clamped plate for $\alpha = 0{\cdot}3$: ∗—∗—∗—∗—∗, $n = 0$; △—△—△—△—△, $n = 1$; ●–●–●–●–●, $n = 2$.

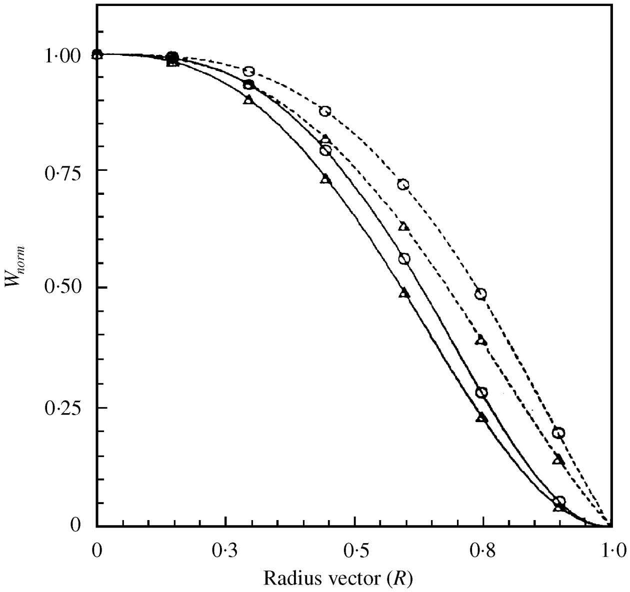

Figure 6. Normalised transverse deflection W_{norm} versus radius vector R for $\nu_\theta = 0{\cdot}3$, $E_\theta/E_r = 5{\cdot}0$, $\alpha = 0{\cdot}3$, SS (-------) and CL (——) plates vibrating in fundamental mode. Mode I (LVT): $K_f = 0{\cdot}00$.

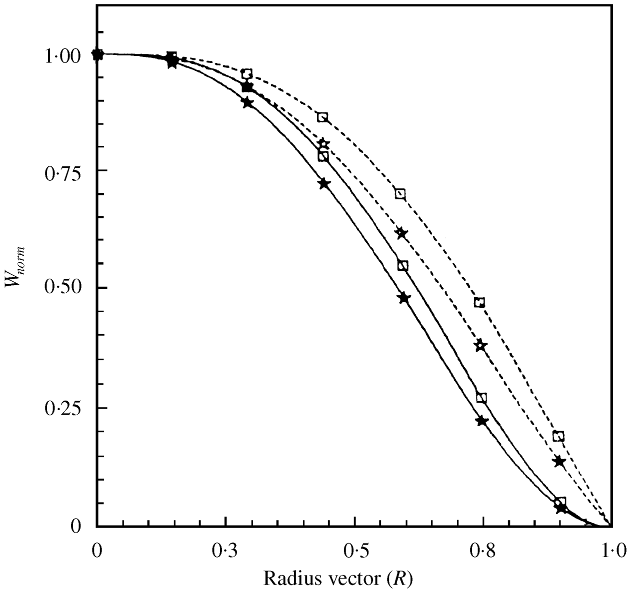

Figure 7. Normalised transverse deflection W_{norm} versus radius vector R for $\nu_\theta = 0{\cdot}3$, $E_\theta/E_r = 5{\cdot}0$, $\alpha = 0{\cdot}3$, SS (-------) and CL (——) plates vibrating in fundamental mode. Mode I (LVT): $K_f = 0{\cdot}01$.

5. DISCUSSION

The frequency equation (7) is solved for various values of rigidity ratio E_θ/E_r ($= 0{\cdot}5, 0{\cdot}75, 1{\cdot}0, 5{\cdot}0$), taper parameter α ($= 0{\cdot}0, \pm 0{\cdot}3$), flexibility parameters K_ϕ ($= 0, 10, 10^{20} \cong \infty$),

Table 4

Comparison of frequency parameter Ω for isotropic circular plate of linearly varying thickness resting on elastic foundation for first two modes: $\nu_\theta = 0{\cdot}33$

	Simply supported edge for $K_f = 10$						Simply supported edge for $K_f = 20$					
α/Ω	Ω_{00}	Ω_{01}	Ω_{10}	Ω_{11}	Ω_{20}	Ω_{21}	Ω_{00}	Ω_{01}	Ω_{10}	Ω_{11}	Ω_{20}	Ω_{21}
	6·30†	33·12†	15·91†	53·93†	29·06†	78·35†	12·62†	44·69†	24·58†	68·13†	39·76†	95·33†
− 0·2	6·2992	33·1186	15·9140	53·8956	29·0618	78·1215	12·5988	44·6540	24·5677	67·9236	39·7539	94·4479
	6·10†	31·53†	15·10†	51·29†	27·46†	74·29†	11·89†	42·38†	23·16†	64·58†	37·47†	90·10†
− 0·1	6·0947	31·5238	15·1015	51·2711	27·4543	74·1936	11·8681	42·3497	23·1497	64·4819	37·4650	89·6064
	5·90†	29·94†	14·29†	48·63†	25·85†	70·26†	11·18†	40·06†	21·74†	61·58†	35·17†	84·94†
0·0	5·8983	29·9220	14·2899	48·6153	25·8429	70·2211	11·1518	40·0218	21·7257	60·9928	35·1626	84·7008
	5·72†	28·35†	13·48†	45·95†	24·23†	66·22†	10·48†	37·74†	20·31†	57·49†	32·86†	79·83†
0·1	5·7111	28·3050	13·4755	45·9231	24·2269	66·1965	10·4532	37·6663	20·2954	57·3569	32·6830	79·6540
	5·54†	26·74†	12·66†	43·26†	22·61†	62·15†	9·80†	35·42†	18·87†	53·92†	30·52†	74·72†
0·2	5·5342	26·6718	12·6580	43·1880	22·6058	62·1101	9·7762	35·2782	18·8590	53·8397	30·5092	74·6511
	Free edge for $K_f = 10$						Free edge for $K_f = 20$					
	—	10·25†		22·77†	6·58†	39·34†	—	10·68†	—	22·96†	7·20†	39·46†
− 0·2	2·9702	10·2576	2·9361	22·7751	6·4865	39·3403	4·2002	10·6880	4·1522	22·9711	7·1154	39·4533
	—	9·92†	—	21·76†	6·39†	37·37†	—	10·39†	—	21·98†	7·07†	37·62†
− 0·1	3·0618	9·9243	3·0429	21·7659	6·3016	37·3639	4·3299	10·3905	4·3033	21·9809	6·9950	37·4897
	—	9·60†	—	20·76†	6·21†	35·39†	—	10·11†	—	21·00†	6·97†	35·53†
0·0	3·1623	9·6044	3·1623	20·7554	6·1391	35·3841	4·4721	10·1116	4·4721	20·9949	6·9057	35·5251
	—	9·30†		19·75†	6·07†	33·41†	—	9·85†	—	20·01†	6·91†	33·57†
0·1	3·2731	9·3008	3·2969	19·7474	6·0042	33·4009	4·6287	9·8557	4·6625	20·0150	6·8539	33·5604
	—	9·00†		18·73†	5·95†	31·43†	—	9·61†	—	19·03†	6·89†	31·61†
0·2	3·3960	9·0174	3·4502	18·7424	5·9032	31·4144	4·8016	9·6291	4·8793	19·0444	6·8475	31·5969

† Values are taken from reference [9].

TABLE 5

Comparison of frequency parameter Ω for uniform isotropic circular plate for $\nu_\theta = 0{\cdot}3$ and different values of K_ϕ

K_ϕ/Ω	Ω_{00}	Ω_{01}	Ω_{02}	Ω_{10}	Ω_{11}	Ω_{12}	Ω_{20}	Ω_{21}	Ω_{22}
					$K = 0$				
	—	9·003†	38·443†	—	20·475†	59·812†	5·358†	35·260†	84·368†
0	—	9·0031	38·4432	—	20·4746	59·8116	5·3583	35·2601	84·3662
	—	13·513†	45·764†	—	26·298†	68·068†	8·050†	41·844†	93·256†
10	—	13·5129	45·7643	2·7735	26·2982	68·0681	8·0497	41·8438	93·2569
		14·539†	48·746†	—	28·126†	72·168†	8·693†	44·483†	98·431†
10^2	—	14·5388	48·7457	—	28·1262	72·1675	8·6933	44·4832	98·4336
	—	14·682†	49·218†	—	28·399†	72·860†	8·785†	44·904†	99·359†
∞	—	14·6820	49·2159	3·0948	28·3988	72·8781	8·7849	44·9031	99·3466
					$K = \infty$				
	4·935†	29·720†	74·156†	13·898†	48·479†	102·772†	25·613†	70·117†	134·290†
0	4·9351	29·7200	74·1560	13·8982	48·4789	102·7733	25·6133	70·1170	134·2978
	8·752†	35·218†	80·685†	18·543†	54·516†	109·650†	30·848†	76·5442†	141·441†
10	8·7519	35·2190	80·6869	18·5438	54·5164	109·6529	30·8481	76·5442	141·4431
	10·019†	39·029†	87·488†	20·858†	59·710†	117·925†	34·226†	83·050†	151·126†
10^2	10·0192	39·0288	87·4900	20·8587	59·7102	117·9353	34·2259	83·0501	151·1094
	10·216†	39·771†	89·103†	21·260†	60·829†	120·077†	34·877†	84·584†	153·830†
∞	10·2158	39·7711	89·1041	21·2604	60·8287	120·0792	34·8770	84·5826	153·8151

† Values are taken from reference [26, 27].

TABLE 6

Comparison of frequency parameter Ω with the exact solution and that obtained by finite element methods

n/Ω_{ni}	SS—plate			CL—plate		
	Ω_{n0}	Ω_{n1}	Ω_{n2}	Ω_{n0}	Ω_{n1}	Ω_{n2}
$n = 0$	4·9352†	29·7222†	74·1938†	10·2159†	39·7766†	89·1708†
	4·9352‡	29·7200‡	74·1961‡	10·2158‡	39·7711‡	89·1041‡
	4·9352	29·7200	74·1961	10·2158	39·7711	89·1041
$n = 1$	13·8983†	48·4867†	102·8465†	21·2611†	60·8441†	120·19581†
	13·8982‡	48·4789‡	102·7734‡	21·2604‡	60·8287‡	120·0793‡
	13·8982	48·4789	102·7734	21·2604	60·8287	120·0793
$n = 2$	25·6145†	70·1404†	134·4588†	34·8799†	84·6236†	154·0564†
	25·6133‡	70·1170‡	134·4529‡	34·8770‡	84·5827†	153·8151‡
	25·6133	70·1170	134·4529	34·8770	84·5827	153·8151

† Taken from reference [34].
‡ Taken from reference [33].

K ($= 0$, 10, $10^{20} \cong \infty$), foundation parameter K_f ($= 0{\cdot}0(0{\cdot}01)0{\cdot}05$) and nodal diameter $n = 0$–2. The numerical results are presented in Tables 1–3 and Figures 2–7. Table 1 gives the values of frequency parameter Ω for different values of rigidity ratio E_θ/E_r, flexibility parameter K_ϕ and foundation parameter $K_f = 0{\cdot}01, 0{\cdot}02$ for $\alpha = \pm 0{\cdot}3$ for first two modes of vibration corresponding to $n = 0$–2, respectively, when the stiffness of translational spring $K = 0{\cdot}0$. The free edge classical boundary condition corresponds to flexibility parameter $K_\phi = 0{\cdot}0$. The frequency parameter Ω is found to increase with the increasing values of rigidity ratio E_θ/E_r as well as flexibility parameter K_ϕ, keeping all other plate parameters fixed. It increases further for higher modes of antisymmetric modes of vibrations. The effect of flexibility parameter K_ϕ and the rigidity ratio are observed to be more pronounced for nodal diameter $n = 2$. It can also be noticed here that the frequency parameter increases with the increase in foundation parameter K_f, other plate parameters being fixed. A similar behaviour is observed in the second mode of vibration except for the fact that the frequency parameter Ω is greater as compared to the foundamental mode. The results show that the frequency parameter Ω for a centrally thinner plate ($\alpha < 0$) is less than that for a centrally thicker plate ($\alpha > 0$) for small values of foundation parameter K_f, but if the nodal diameter n, or flexibility parameter K_ϕ or rigidity ratio or mode of vibration increases, the effect of taper on frequency parameters is observed to be just the reverse. Table 2 presents the frequency parameter Ω for $K = 10$, other plate parameter values being same as in Table 1. The table shows that the frequency parameter is further increased with the increase in stiffness of the translational spring parameter K. The frequency parameter Ω for clamped plate ($K_\phi = 10^{20}$ and $K = 10^{20}$) is greater than that for the simply supported plate ($K_\phi = 0$ and $K = 10^{20}$), presented in Table 3. The effect of elastic foundation on frequency parameter for the simply supported plate is greater than that for the clamped plate, but lesser than that for the free edge plate.

Figure 2 is the plot for frequency parameter Ω versus foundation parameter K_f for different values of rigidity ratio E_θ/E_r ($= 0{\cdot}75$, 1, 5·0) for a centrally thinner ($\alpha = -0{\cdot}3$) simply supported (SS) plate. The graph shows that the frequency parameter Ω increases with foundation parameter K_f. The effect of rigidity ratio on frequency parameter Ω is quite appreciable for increasing values of nodal diameter n. The rate of increase of frequency parameter with the increase in foundation parameter for a radially stiffened plate ($E_\theta/E_r = 0{\cdot}75$) is more than that for a circumferentially stiffened plate ($E_\theta/E_r = 5{\cdot}0$). The frequency parameter Ω increases almost linearly for $n = 2$. Figure 3 is the plot for frequency parameter Ω versus foundation parameter K_f for a centrally thicker plate ($\alpha = 0{\cdot}3$) for the simply supported edge vibrating in fundamental mode for nodal diameters $n = 0$–2. The graphs 2 and 3 show that the rate of increase of frequency parameter for a centrally thinner plate is less than that for a centrally thicker plate. Figures 4 and 5 show the frequency parameter Ω for different values of foundation parameter K_f for a clamped (CL) plate vibating in fundamental mode for $n = 0$–2 for $\alpha = -0{\cdot}3$ and $0{\cdot}3$ respectively. The frequency parameter Ω increases with the increase in K_f and is greater than that for a simply supported plate, other plate parameters being fixed. The effect of increase of nodal diameter on frequency parameter for a radially stiffened plate ($E_\theta < E_r$) is less than for a circumferencially stiffened clamped plate ($E_\theta > E_r$). The normalized transverse deflection W_{norm} is presented in graphs 6 and 7 for simply supported (----) and clamped (——) plates; taper parameter $\alpha = 0{\cdot}3$ for foundation parameter $K_f = 0{\cdot}00$ and $0{\cdot}01$ for LVT plates. The transverse deflection is maximum at the centre for both the plates. It is seen that transverse deflection for axisymmetric mode ($n = 0$) is greater than that for antisymmetric mode ($n = 1$). The figures also show that the transverse deflection decreases as the foundation parameter K_f increases. Further, it can be noticed here that the transverse deflection for the simply supported plate is greater than that for the clamped plate for the corresponding values of plate parameters.

A comparison of our results with those available in the literature by Laura *et al.* [9], Pardoen [34] and Azimi [18, 26] shows an excellent agreement and is reported in Tables 4–6 respectively.

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APPENDIX A: NOMENCLATURE

r, θ	polar co-ordinates of a point in the mid-plane of the plate
a	radius of the circular plate

R	r/a, non-dimensional radius vector
t	time
w	$w(r, \theta, t)$ transverse displacement function
$\bar{W}(R, \theta)$	$W(R)\cos n\theta$
W_{norm}	W/W_{max}
E_r, E_θ	Young's moduli of elasticity along r and θ directions respectively
$G_{r\theta}$	shear modulus
v_r, v_θ	the Poisson ratio defined as strain in tangential and radial directions respectively
h_0	non-dimensional thickness of the plate at the centre
h	$h(r)$, thickness of the plate
W_i	shape function
W'_i, W''_i	$\dfrac{dW_i}{dR}, \dfrac{d^2 W_i}{dR^2}$
α	taper constant
α_i, β_i	constants
D_r, D_θ	$E_r h^3/12(1 - v_r v_\theta)$, $E_\theta h^3/12(1 - v_r v_\theta)$, flexural rigidities along r and θ directions respectively
$D_{r\theta}$	$G_{r\theta} h^3/12$, shear rigidity
D_k	$4D_{r\theta}$
D_{r0}	$E_r h_0^3/12(1 - v_r v_\theta)$
D_{k0}	D_k/D_{r0}
σ_r, σ_θ, $\sigma_{r\theta}$	radial, tangential and shear stresses
M_r, M_θ, $M_{r\theta}$	radial, tangential and shear moments
T_{max}	maximum kinetic energy
U_{max}	maximum strain energy of the plate
k	stiffness of the spring against translation
k_ϕ	stiffness of the spring against rotation
k_f	foundation modulus
K, K_ϕ	$a^3 k/D_{r\theta}$, $a\, k_\phi/D_{r\theta}$, flexibility of the springs
K_f	$a^4 k_f/D_{r\theta}$, foundation modulus
ρ	density of the plate material
ω	circular frequency in rad/s
Ω^2	$a^4 \rho \omega^2 h_0/D_{r\theta}$, frequency parameter
A_i	constants
Ω_{nj}	frequency parameter in jth mode